Grade 7/8 Math Circles

Nov 7/8/9/10, 2022

Induction - Solutions

1. True or False? Provide an explanation.
   (i) For all $n \in \mathbb{N}$, $n < n^2$.
   (ii) For all $n \in \mathbb{N}$, $n + 2 \leq 3^n$.
   (iii) For all $n \in \mathbb{N}$, $\frac{1}{3}(n+1)(n+2) \in \mathbb{N}$.
   (iv) For all $n \in \mathbb{N}$, $n^2 - 1 = (n-1)(n+1)$.

Solution:
(i) False. When $n = 1$, $1 < 1$ is false.
(ii) True. When $n = 1$, $1 + 2 \leq 3^1$ is true. As n increases by 1, the left side increases by 1 and the right side increases by a factor of 3. Therefore, the left side will always be less than or equal to the right.
(iii) False. When $n = 3$, $\frac{1}{3}(n+1)(n+2) = \frac{20}{3} \notin \mathbb{N}$.
(iv) True. This follows from the distributive property of whole numbers:

$$\begin{aligned}(n-1)(n+1) &= n(n+1) - (n+1)\\ &= n^2 + n - n - 1\\ &= n^2 - 1\end{aligned}$$

2. For $n \in \mathbb{N}$, let F_n be the nth Fibonacci number. That is, let $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$. Prove that

$$F_{n+2} = 1 + F_1 + F_2 + ... + F_n$$

for all $n \in \mathbb{N}$.

Solution: We prove this by induction on n. Let $P(n)$ be the statement

$$F_{n+2} = 1 + F_1 + F_2 + ... + F_n$$

- Base Case: When $n = 1$, $1 + F_1 = 2 = F_3$ therefore $P(1)$ is true.
- Inductive Hypothesis: Let $k \in \mathbb{N}$ and assume $P(k)$ is true: $F_{k+2} = 1 + F_1 + ... + F_k$.
- Inductive Step: Using the recursive property of the Fibonacci numbers, and the above inductive hypothesis,

$$\begin{aligned} F_{k+3} &= F_{k+2} + F_{k+1} \\ &= (1 + F_1 + ... + F_k) + F_{k+1} \\ &= 1 + F_1 + F_2 + ... + F_{k+1} \end{aligned}$$

  therefore $P(k+1)$ is true.
- Conclusion: By the Principle of Mathematical Induction, we can conclude that $P(n)$ is true for all $n \in \mathbb{N}$.

3. Use induction to prove the Pigeonhole Principle: 'For all $n \in \mathbb{N}$, if $n+1$ pigeons occupy n holes, then some hole must have at least 2 pigeons.'

Solution: Let $P(n)$ be: 'if $n + 1$ pigeons occupy n holes, then some hole must have at least 2 pigeons'.

- Base Case: If 2 pigeons occupy 1 hole, then the hole must have at least 2 pigeons. Thus, $P(1)$ is true.
- Inductive Hypothesis: Let $k \in \mathbb{N}$, and assume that $P(k)$ is true: if $k + 1$ pigeons occupy k holes, then some hole must have at least 2 pigeons.
- Inductive Step: Suppose $k + 2$ pigeons occupy $k + 1$ holes. Consider the first hole:
  - If it has at least 2 pigeons, then we have found a hole with at least 2 pigeons.
  - Otherwise, there are at least $k+1$ pigeons remaining which occupy k holes. By the inductive hypothesis, there is a hole with at least 2 pigeons.

  Therefore, there is a hole with at least 2 pigeons. Thus, $P(k + 1)$ is true.
- Conclusion: By the Principle of Mathematical Induction, we can conclude that $P(n)$ is true for all $n \in \mathbb{N}$.

4. The following proof shows that every horse has the same colour. Find the mistake:
   Let $P(n)$ be the statement 'in any group of n horses, all horses have the same colour'.

   - Base Case: If there is only one horse in a group, then there is only one colour of horse in that group. Therefore $P(1)$ is true.
   - Inductive Hypothesis: Let k be a natural number and assume $P(k)$ is true. That is, in any group of k horses, all horses have the same colour.
   - Inductive Step: Consider any group of $k+1$ horses. Remove a horse from the group. By the inductive hypothesis, all other horses must have the same colour. Add the horse back to the group and remove a different horse. Once again, all horses have the same colour. Therefore, the horse we initially removed has the same colour as the other k horses. We can conclude that all horses in this group have the same colour.
   - Conclusion: By the Principle of Mathematical Induction, for all $k \in \mathbb{N}$, any group of k horses has the same colour.

   Since there are a finite number of horses in the world, all horses have the same colour.

> *Solution*: The mistake occurs in the inductive step. Consider a group of horses, which include H_1 (horse 1), H_2 (horse 2), and R (the rest of the horses in the group). We can apply the inductive hypothesis to find that
>
> - H_2 and all horses in R have the same colour
> - H_1 and all horses in R have the same colour
>
> The inductive step then claims that H_1 and H_2 must have the same colour as the horses in R. However, if R is empty (there is only H_1 and H_2), then there is no way of concluding that H_1 and H_2 have the same colour.

5. Prove that $1 + \frac{n}{2} \leq 1.5^n$ for all $n \in \mathbb{N}$.

> *Solution*: Once again, we use induction on n. Let $P(n)$ be the statement $1 + \frac{n}{2} \leq 1.5^n$.
>
> - Base Case: when $n = 1$, $1.5 \leq 1.5$ is true, therefore $P(1)$ is true.
> - Inductive Hypothesis: Let $k \in \mathbb{N}$ and assume $P(k)$ is true:
>
> $$1 + \frac{k}{2} \leq 1.5^k$$

- Inductive Step: Using the inductive hypothesis, we have

$$1.5^{k+1} = 1.5 \times 1.5^k \geq 1.5 \times (1 + 0.5k) = 1.5 + 0.75k$$

We can then use this to prove that $P(k+1)$ is true:

$$1 + \frac{k+1}{2} = 1.5 + 0.5k < 1.5 + 0.75k \leq 1.5^{k+1}$$

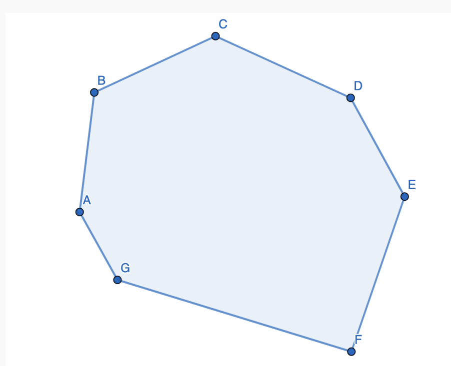
Therefore $1 + \frac{k+1}{2} \leq 1.5^{k+1}$.

- Conclusion: By the Principle of Mathematical Induction, $1 + \frac{n}{2} \leq 1.5^n$ for all $n \in \mathbb{N}$.

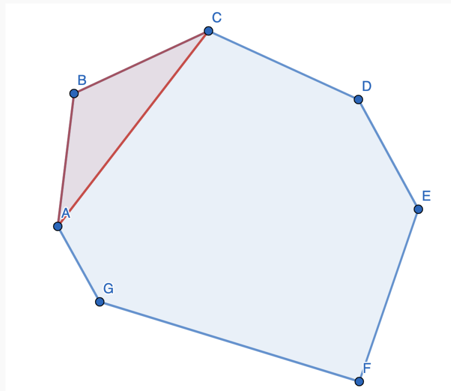
6. For all natural numbers $n \geq 3$, prove that the sum of the interior angles of a polygon with n sides is $180^\circ \times (n-2)$.

Solution: We use induction on n, but our base case is no longer $n = 1$. In the Principle of Mathematical Induction, the base case is the first statement to be proven true (e.g., the first domino to fall over). Therefore, if we use $n = 3$ as our base case, induction can still be applied to show that the statement is true for all natural numbers $n \geq 3$.

- Base Case: When $n = 3$, a polygon with n sides is just a triangle. The sum of the interior angles in a triangle is 180°. Therefore the base case is true.
- Inductive Hypothesis: Let $3 \leq k \in \mathbb{N}$ and assume that the sum of the interior angles of any polygon with k sides is $180^\circ \times (k-2)$.
- Inductive Step: Consider a polygon with $k+1$ sides. Pick three adjacent vertices to form a triangle. For example, if our polygon is



Then we can pick A, B, and C to form a triangle. Our $(k+1)$-sided polygon can now be separated into a triangle and an k-sided polygon.



The sum of the interior angles of the triangle is $180°$ and the sum of the interior angles of the k-sided polygon is $180° \times (k-2)$ (inductive hypothesis). Therefore, the sum of the interior angles of the $(k+1)$-sided polygon is

$$180° + 180° \times (k-2) = 180° \times (k-1)$$

so the statement is true for $k+1$.

- Conclusion: By the Principle of Mathematical Induction, we can conclude that for any polygon with n sides (where $3 \leq n$), the sum of its interior angles is $180° \times (n-2)$.

7. Prove that any natural number can be written as the sum of (one or more) distinct powers of 2 (the powers of 2 are $2^0 = 1$, $2^1 = 2$, $2^2 = 4$, $2^3 = 8$, $2^4 = 16$, ...). For example, $19 = 16 + 2 + 1 = 2^4 + 2^1 + 2^0$.

Solution: We do this by strong induction on n. Let $P(n)$ be the statement 'n can be written as a sum of distinct powers of 2'.

- Base Case: $1 = 2^0$ therefore $P(1)$ is true.
- Inductive Hypothesis: let $k \in \mathbb{N}$ and assume $P(k')$ is true for all $k' \leq k$.
- Inductive Step: Consider the largest power of 2 that is at most $k+1$. That is, define x such that

$$2^x \leq k + 1 < 2^{x+1}$$

Then $(k + 1) - 2^x$ is at most k. Therefore, $(k + 1) - 2^x$ can be written as the sum of distinct powers of two. Let L the list of (distinct) powers of 2 which add up to $(k + 1) - 2^x$ (note L could be empty). Notice

$$(k + 1) - 2^x < 2^{x+1} - 2^x = 2^x$$

therefore all powers of 2 in L are less than 2^x. Adding all the powers of 2 in L with 2^x, we obtain $k + 1$. This provides us with a way of writing $k + 1$ as the sum of distinct powers of 2. So $P(k + 1)$ is true.

- Conclusion: By Strong Induction, we can conclude that any natural number can be written as the sum of distinct powers of 2.

For example: suppose we are writing 169 as the sum of distinct powers of 2. The greatest power of 2 that is less than 169 is 128, so we start by writing

$$169 = 128 + 41$$

Next, the greatest power of 2 that is less than 41 is 32 so we continue with

$$169 = 128 + 32 + 9$$

Finally, we see that $9 = 8 + 1$ so we obtain

$$169 = 128 + 32 + 8 + 1 = 2^7 + 2^5 + 2^3 + 2^0$$

8. Bonus Problem: A group of 1000 people lives on a mysterious island. Every person has either blue eyes or green eyers. None of them know their eye colour, and they are forbidden to discuss the topic; thus, each resident can (and does) see the eye colours of all other residents, but has no way of discovering their own (there are no reflective surfaces). If someone does discover their own eye colour, then they must leave the island the following day, in front of the entire the village. Assume every person on the island is highly logical, and knows that everyone else is highly logical.

   Of the 1000 islanders, it turns out that 100 of them have blue eyes and the rest have green eyes, although the islanders are not initially aware of these statistics (each of them can of course only see the other 999 people). One day, a foreigner visits to the island, winning the complete trust of the islanders. Not knowing the customs, the foreigner makes the mistake of mentioning eye colour in their address, remarking "how unusual it is to see another blue-eyed person like myself in this region of the world".

   What effect, if anything, do the foreigner's words have on the residents of the island?

   (Adapted from a post on Terrence Tao's blog)

   Solution: Exactly 100 days after the foreigner's words, every person with blue eyes will leave the island.

   In order to better understand the problem, consider a smaller version of the problem: there are 5 islanders, 3 of whom have blue eyes. Let day 0 be the day of the foreigner's address.

   - Day 1: Since each islander already knows there is someone with blue eyes, no one leaves the island. Therefore, every islander knows that there are at least 2 people with blue eyes (otherwise someone would leave the island).
   - Day 2: Every islander can see at least 2 people with blue eyes so no one leaves the island. However, each islander now deduces that there at least 3 people with blue eyes (otherwise someone would leave the island). Hence, the islanders with blue eyes realize that they have blue eyes.
   - Having made their realization, the 3 islanders with blue eyes leave the island.

In our more general case, we will use induction to prove the statement $P(n)$: if there are exactly n people on the island with blue eyes, then all n of them will leave the island on day n. Once again, assume day 0 is the day of the foreigner's address.

- Base Case: if there is only 1 person with blue eyes, then they will realize that the foreigner is referring to them. Therefore, they will leave the island the next day. Hence, $P(1)$ is true.
- Inductive Hypothesis: let $k \in \mathbb{N}$ and assume $P(k)$: if there are exactly k people on the island with blue eyes, then they will all leave on day k.
- Inductive Step: Suppose there are $k+1$ people on the island with blue eyes. Since each person with blue eyes can see k blue-eyed islanders, no one will leave on day k. But all of the islanders are highly logical, and understand that there must be more than k people with blue eyes (otherwise they would have left the island). Therefore, the $k+1$ blue-eyed islanders realize that they have blue eyes, and will leave on day $k+1$.
- Conclusion: By the Principle of Mathematical Induction, $P(n)$ is true for all $n \in \mathbb{N}$.

Therefore, after 100 days, the 100 people with blue eyes will leave the island.

The key concept here is **common knowledge**: there is a difference between

- All islanders know that there are at least 99 people with blue eyes.
- All islanders know that: 'All islanders know that there are at least 99 people with blue eyes'.